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The Lax Equivalence Theorem for Linear, Inhomogeneous Equations in L^2 Spaces

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1. INTRODUCTION

The original Lax Equivalence Theorem, Lax and Richtmyer [4], Morton and Richtmyer [6], was subsequently generalised in various directions. For instance, Thompson, [7], extended the result to inhomogeneous equations in which the inhomogeneous term, f, is piecewise continuous on $[0, T]$, the time interval of interest. Also, in Thompson's paper and in Ansorge [1], semi-linear initial value problems are treated. Recently there has been work done on equipping the stability estimates, etc. with orders (powers of the time step Δt), see, e.g., Butzer and Weis [2], Butzer *et al.* [3].

This paper extends the result to the case of inhomogeneous terms which are merely Lebesgue square integrable on $[0, T]$. The motivation for this extension comes from the use of finite difference approximations in the study of optimal control of systems governed by partial differential equations. To guarantee the existence of optimal solutions one has to work in L^2 spaces, Lions [5].

2. THE DIFFERENCE APPROXIMATION

In this paper it is shown that under the conditions of the Lax Equivalence Theorem (i.e., that $\{C(\Delta t)\}$ is a consistent and stable approximation to a well-posed initial value problem), the solution of the difference equations converges to the exact solution of the differential equation, when the inhomogeneous term, f, belongs to $B_2 = L^2([0, T]: X)$, the (Bochner) space of functions $g(\cdot)$, with values in the Banach space X and $\|g(t)\|^2$ square integrable on $[0, T]$. The case of f continuous on $[0, T]$ is proved in Thompson [7], and will be assumed here.

The differential equation is

$$\frac{dx}{dt} = Ax + f, \qquad x(0) = x_0 \in X, \tag{1}$$

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where X is a Banach space and $f \in B_2$. A is a closed, densely defined linear operator on X, generating a strongly continuous semi-group $E(t)$. If f is continuous on $[0, T]$, i.e., $f \in C([0, T]; X)$, the difference equations take the form

$$x^{k+1} = C(\Delta_j t)x^k + \Delta_j t f(k\Delta_j t), \qquad x^0 = x_0.$$

However, if f is not continuous but belongs to B_2, $\|f(t)\|$ may be unbounded on a set of measure zero in $[0, T]$. So we take the difference equations in the form

$$x^{k+1} = C(\Delta_j t)x^k + \Delta_j t f^k, \qquad x^0 = x_0, \tag{2}$$

where f^k is defined by

$$f^k = \frac{1}{\Delta_j t} \int_{k\Delta_j t}^{\overline{k+1}\Delta_j t} f(t)\, dt. \tag{3}$$

The generalised solution of the differential equation (1) is

$$x(t) = E(t)x_0 + \int_0^t E(t-s) f(s)\, ds,$$

and the solution of the difference equations is

$$x^n = C^n(\Delta_j t)x_0 + \Delta_j t \sum_{k=0}^{n-1} (C(\Delta_j t))^{n-k-1} f^k.$$

The assumptions of consistency and stability for the approximation $\{C(\Delta_j t)\}$ imply that $C^{n_j}(\Delta_j t)x_0 \to E(t)x_0$ as $\Delta_j t \to 0$ and $n_j \Delta_j t \to t$. So it only needs to be shown that

$$\Delta_j t \sum_{k=0}^{n-1} (C(\Delta_j t))^{n-k-1} f^k$$

approximates the solution of (1) when $x_0 = 0$, for $f \in B_2$.

Make the definitions

$$(\tilde{G}_j f)(n\Delta_j t) = \Delta_j t \sum_{k=0}^{n-1} (C(\Delta_j t))^{n-k-1} f(k\Delta_j t)$$

for $f \in C([0, T]; X)$ and

$$(G_j f)(n\Delta_j t) = \Delta_j t \sum_{k=0}^{n-1} (C(\Delta_j t))^{n-k-1} f^k$$

for $f \in B_2$. It is proved in Thompson's paper that $(\tilde{G}_j f)(n\Delta_j t) \to x(t)$ as $\Delta_j t \to 0$, $n\Delta_j t \to t$, where $x(t)$ is the solution of (1), when f is continuous. We have to show that $(G_j f)(n\,\Delta_j t) \to x(t)$ when f belongs to B_2.

3. Convergence Proof

Since the initial value problem $\dot{x} = Ax$ is assumed well-posed, there exists $K > 0$ such that $\|E(t)\| \leqslant K$ for $0 \leqslant t \leqslant T$. The approximation is assumed to be stable so there is a $J > 0$ such that $\|C^n(\Delta_j t)\| \leqslant J$ for $0 \leqslant \Delta_j t \leqslant \tau$, $0 \leqslant n\Delta_j t \leqslant T$. Now pick $\varepsilon > 0$.

For $f \in B_2$ there is an $f_\varepsilon \in C([0, T]; X)$ such that

$$\|f - f_\varepsilon\|_{B_2} < \min\left(\frac{\varepsilon}{4K\sqrt{T}}, \frac{\varepsilon}{4J\sqrt{T}}\right) \tag{4}$$

since the continuous functions are dense in the L^2 functions. The norm on B_2 is defined by

$$\|f\|_{B_2} = \left\{\int_0^T \|f(t)\|^2\, dt\right\}^{1/2},$$

$\|f(t)\|$ being the norm of $f(t) \in X$. Denote the solution of (1) with f replaced by f_ε as x_ε. Then

$$x(t) - x_\varepsilon(t) = \int_0^t E(t-s)(f(s) - f_\varepsilon(s))\, ds$$

and by the Schwarz inequality and (4) (noting that $\int_0^t \|E(t-s)\|^2\, ds \leqslant K^2 T$)

$$\|x(t) - x_\varepsilon(t)\| \leqslant K\sqrt{T}\,\|f - f_\varepsilon\|_{B_2} \leqslant \varepsilon/4.$$

By assumption the Lax theorem is true for inhomogeneous terms which are continuous on $[0, T]$, when $\tilde{G}_j$ is taken as the approximate solution operator. So there exists $\delta_1 > 0$ such that

$$0 < \Delta_j t \leqslant \tau, \qquad n\Delta_j t \in [0, T] \text{ with } |t - n\Delta_j t| < \delta_1$$

implies

$$\|x_\varepsilon(t) - (\tilde{G}_j f_\varepsilon)(n\Delta_j t)\| \leqslant \varepsilon/4 \tag{5}$$

uniformly in t, [7]. Now from (3)

$$f_\varepsilon(k\Delta_j t) - f_\varepsilon^k = \frac{1}{\Delta_j t}\int_{k\Delta_j t}^{\overline{k+1}\,\Delta_j t} (f_\varepsilon(k\,\Delta_j t) - f_\varepsilon(t))\, dt,$$

and since f_ϵ is continuous on $[0, T]$, it is uniformly continuous on $[0, T]$, so there is a $\delta_2 > 0$ such that for all s, t in $[0, T]$, $|s - t| < \delta_2$ implies $\|f_\epsilon(s) - f_\epsilon(t)\| \leqslant \varepsilon/4JT$. Hence, if $\Delta_j t \leqslant \delta_2$

$$\|f_\epsilon(k\Delta_j t) - f_\epsilon^k\| \leqslant \varepsilon/4JT.$$

Therefore the norm of

$$(\tilde{G}_j f_\epsilon)(n\Delta_j t) - (G_j f_\epsilon)(n\Delta_j t)$$
$$= \Delta_j t \sum_{k=0}^{n-1} (C(\Delta_j t))^{n-k-1}\{f_\epsilon(k\Delta_j t) - f_\epsilon^k\}$$

is uniformly bounded by $\varepsilon/4$.

By the Schwarz inequality for sums,

$$\|(G_j f_\epsilon)(n\Delta_j t) - (G_j f)(n\Delta_j t)\|$$
$$\leqslant \Delta_j t \left\{\sum_{k=0}^{n-1} \|(C(\Delta_j t))^{n-k-1}\|^2\right\}^{1/2}$$
$$\times \left\{\sum_{k=0}^{n-1} \|f_\epsilon^k - f^k\|^2\right\}^{1/2}. \tag{6}$$

Again the Schwarz inequality gives (cf. (3))

$$\|f_\epsilon^k - f^k\|^2 \leqslant \frac{1}{\Delta_j t} \int_{k\Delta_j t}^{\overline{k+1}\,\Delta_j t} \|f_\epsilon(t) - f(t)\|^2\, dt,$$

and thus

$$\sum_{k=0}^{n-1} \|f_\epsilon^k - f^k\|^2 \leqslant \frac{1}{\Delta_j t} \|f - f_\epsilon\|_{B_2}^2.$$

Since for the other term in (6) we have

$$\sum_{k=0}^{n-1} \|(C(\Delta_j t))^{n-k-1}\|^2 \leqslant J^2 n,$$

one obtains, noting $n\Delta_j t \leqslant T$ and (4),

$$\|(G_j f_\epsilon)(n\Delta_j t) - (G_j f)(n\Delta_j t)\| \leqslant \Delta_j tJ\sqrt{n}\,\frac{1}{\sqrt{\Delta_j t}}\|f - f_\epsilon\|_{B_2} \leqslant \varepsilon/4.$$

Now take $0 < \Delta_j t \leqslant \min(\tau, \delta_2)$ and $|t - n\Delta_j t| < \delta_1$, so that repeated application of the triangle inequality yields

$$\begin{aligned}\|x(t) - (G_j f)(n\Delta_j t)\| \\ &\leqslant \|x(t) - x_\epsilon(t)\| + \|x_\epsilon(t) - (\tilde{G}_j f_\epsilon)(n\Delta_j t)\| \\ &\quad + \|(\tilde{G}_j f_\epsilon)(n\Delta_j t) - (G_j f_\epsilon)(n\Delta_j t)\| \\ &\quad + \|(G_j f_\epsilon)(n\Delta_j t) - (G_j f)(n\Delta_j t)\| \leqslant \varepsilon.\end{aligned}$$

Since this is true for any $\varepsilon > 0$,

$$(G_j f)(n\Delta_j t) \to x(t),$$

and the Lax theorem is proved for the case of f belonging to $L^2([0, T]; X)$. Note that the convergence is uniform in t, since (5) holds uniformly in t.

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